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Theory and Methodology

Optimal strategies for general price–quality decision models of new products with learning production costs [1]

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Abstract

The question of product quality permeates every level of business and is becoming crucial for the survival of modern manufacturing firms in automotive and high-tech industries. In this paper, we deal with the optimal price and quality policies for the introduction of a new product. On the supply side, the firm wants to determine the unit price and quality level over time given that unit cost declines along a learning curve, and increases if quality is made greater. On the demand side, dynamic demand is related to price and quality, as well as to cumulative sales (which represent diffusion and saturation effects). We will model this problem in a general framework that includes several previous results as special cases.

By applying the maximum principle, we will derive the optimal price and quality policies and discover the interactions between these two major strategic marketing instruments, and the diffusion process. Several fundamental theoretical results will be established for the model. Under certain specified conditions higher prices do imply higher quality, and under other conditions the optimal price declines over time while the product quality improves. To illustrate these results, the theoretical results are applied to two specific cases: the first one is a simple nonseparable demand growth function in price and quality, the other is a separable demand function of price and quality.

Keywords: Quality; Pricing; Production; Marketing; Optimal control; New products; Diffusion models; Learning curve

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1. Introduction

In the last several years, there has been a considerable body of literature in the areas of optimal pricing or advertising on new product decision models with learning production costs. For example, Bayus (1992), Kalish (1983, 1985), and Robinson and Lakhani (1975) provide general results for monopolistic pricing. Dockner and Jorgensen (1988a), Horsky and Simon (1983), and Teng and Thompson (1983, 1985) analyze dynamic advertising policies for a monopolist. Clarke and Dolan (1984), Dockner and Jorgensen (1988b), Eliashberg and Jeuland (1986), Erickson (1992), Horsky and Mate (1988), Rao and Bass (1985), and Thompson and Teng

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(1984) deal with the dynamics of duopolistic and oligopolistic price or advertising competition. While product quality did not receive much attention in literature until recently, it has become crucial for the survival of modern manufacturing firms in industries such as automobiles, television sets, steel, computer chips, and other high-tech products. Hence, the purpose of this paper is to establish a general quality–price decision model which incorporates learning curve phenomenon on the production side, and diffusion effect on the demand side. We will explore the optimal quality and pricing policies, and characterize the conditions for which higher prices imply either higher or lower quality.

We will restrict the work here to deal with a monopolistic market where there is no direct competition. This assumption is appropriate and relevant, at least for a new product during a certain period of time before a second firm enters the market, or in a situation in which there is patent protection. In this paper, we also define product quality as some attribute which consumers can easily identify. The reader is referred to articles by Feichtinger, Luhmer and Sorger (1988), Shapiro (1982), Smallwood and Conlisk (1979), and Tellis and Wernerfelt (1987) who analyze markets in which consumers are imperfectly informed about product quality or pricing. Other recent papers related to price and/or quality were written by Balachander and Srinivasan (1994), Gaimon (1988, 1996), Ghosh and Narasimhan (1993), Moorthy (1988), Purohit (1994), Sheshinski (1976), and others. In contrast to those papers, we present long-term price–quality decision models in which the sales rate (demand) is a dynamic function of price and quality, as well as cumulative sales. In other words, we assume that an increase in product quality increases sales directly, and reduces production cost indirectly through the learning curve effect.

2. Model formulation and solution

We begin the analysis by stating the model with as few assumptions as possible. This allows us to derive and examine the most general and robust conclusions. Subsequently, we will consider more specific cases for which we obtain stronger results.

2.1. Quality-related cost and learning production cost

There are two basic kinds of quality: quality of design and quality of conformance (e.g., Fine, 1986). Design quality refers to features, styling, and other attributes which are observable and communicable to consumers. Conformance quality refers to the degree to which a manufactured product conforms to the design for the product, which is not immediately known to consumers. We define quality here as a 'search' attribute (such as quality of design) of which consumers prefer more to less, and can easily identify by search. Hence, we assume (as well as in Moorthy (1988) and Balachander and Srinivasan (1994)) that the quality-related cost for a manufacturer, $c_1(q)$, is a strictly increasing convex function of its quality level q. That is, $\mathrm{d}c_1(q)/\mathrm{d}q > 0$ and $\mathrm{d}^2c_1(q)/\mathrm{d}q^2 > 0$. For instance, a personal computer with a Pentium processor is more preferable to consumers but more costly to producers than one with a 486 or 386 processor. Similarly, a 2-head VCR is less desirable but cheaper than a 4-head VCR. Moreover, this assumption may also apply to traditional companies that rely on inspection to improve quality by weeding out defects. Furthermore, the conformance quality here is not irreversible. For example, Hitachi lowers its VCR price by providing a no-frill 2-head model instead of a 4-head VCR. Similarly, a company through inspection to improve its quality may reduce its quality by reducing its inspection budget.

Arrow (1962), Rosen (1972), and others have observed that the unit production cost of a product declines by a factor of 10 to 50 percent each time the accumulated production volume doubles, due to learning by doing. Hence, the marginal cost of production at time t (denoted by $c_2(t)$) depends on cumulative sales at time t (denoted by $x(t)$) such that the marginal cost decreases with increasing cumulative output (experience), $\mathrm{d}c_2(x)/\mathrm{d}x < 0$.

For simplicity, in this paper we assume that the learning phenomenon is the above volume-based learning only. Thus, we will now assume that the total unit cost c is a function of quality level (q), and experience (or cumulative sales x), that is $c = c(q,x)$, which satisfies the following conditions:

$$c_q > 0, \quad c_{qq} > 0, \quad c_x < 0, \quad \text{and} \quad c_{qx} = 0,$$

where a subscript on a variable denotes partial differentiation with respect to that variable throughout the paper.

2.2. Dynamic diffusion demand

In this paper, sales at time t is a function of price, denoted by $p(t)$, and quality level $q(t)$, as well as the cumulative sales $x(t)$. That is, the sales function $f = f(p(t), q(t), x(t))$. We assume that the demand growth function is twice differentiable and increases with quality and decreases with price. Therefore, we have

$$f_p < 0, \quad f_q > 0, \quad \text{and} \quad f_{pq} = f_{qp}.$$

Bass (1969) used the following new product sales function, denoted by

$$f = \frac{\mathrm{d}x(t)}{\mathrm{d}t} = m(M - x) + n(M - x)x,$$

which is a special form of the general sales function discussed here (where M is the total number of potential purchasers over the life cycle of the product, m is the innovation coefficient and n is the imitation coefficient). Bass (1969) ran regressions on sales for consumer durable goods during the first years of their introduction. These regression results are remarkably good. Since then, the Bass diffusion model has become the standard form of modeling the sales of new durable products. The reader is referred to the recent review papers by Feichtinger, Hartl and Sethi (1994), and Mahajan, Muller and Bass (1990).

2.3. Mathematical model

To define the general price–quality decision model, we first state the following notation:

T = terminal time of the planning horizon,
r = discount rate,
$p(t)$ = unit price at time t (decision variable),
$q(t)$ = quality level at time t (decision variable),
$x(t)$ = cumulative sales volume at time t (state variable),
$c(q, x)$ = total cost per unit,
$\dot{x}(t)$ = $\mathrm{d}x(t)/\mathrm{d}t = f(p, q, x)$ = sales rate at time t.

We assume that the firm wants to maximize its total present value of profit over the planning horizon with length T. Thus, the objective function for the firm is

$$\max_{p(t),q(t)} J = \int_0^T e^{-rt}[p(t) - c(q(t),x(t))]\,\dot{x}(t)\,\mathrm{d}t, \tag{1}$$

subject to

$$\dot{x}(t) = f(p(t),q(t),x(t)). \tag{2}$$

This is a dynamic optimization problem, which is a generalization of many of the previously published studies (Robinson and Lakhani (1975), Kalish (1983), and others). For convenience, a dot above a variable denotes the first derivative with respect to time.

2.4. Optimal solution

To solve the problem, we apply the maximum principle (Sethi and Thompson, 1981). We formulate the current-value Hamiltonian as follows:

$$H = [p(t) - c(q(t),x(t)) + \alpha(t)] f(p(t),q(t),x(t)), \tag{3}$$

where $\alpha(t)$ is the current-value adjoint variable (shadow price of $\dot{x}$) which satisfies the following differential equation:

$$\dot{\alpha}(t) = r\alpha(t) - H_x = r\alpha + c_x f - (p - c + \alpha) f_x, \tag{4}$$

with the transversality condition at $t = T$

$$\alpha(t) = 0. \tag{5}$$

In (4) and what follows we shall omit the function arguments where there is no confusion, to improve clarity. A simple economic interpretation of the current-value Hamiltonian, H, is as follows: $\alpha(t)$ stands for the future benefit (at time t) of having one more unit produced. Thus, the current-value Hamiltonian is the sum of the current profit, $(p - c)f$, and the future benefit, αf. In short, H represents the instantaneous total profit of the firm at time t.

The following necessary conditions hold for an optimal solution:

$$H_p = 0 \Rightarrow p - c + \alpha = -f/f_p, \tag{6}$$

$$H_q = 0 \Rightarrow p - c + \alpha = c_q f/f_q, \tag{7}$$

and the second-order conditions for H-maximization are:

$$H_{pp} < 0 \Rightarrow 2 - ff_{pp}/f_p^2 > 0, \tag{8}$$

$$H_{qq} < 0 \Rightarrow c_{qq} f - (2f_q^2 - ff_{qq})/f_p > 0, \tag{9}$$

and $H_{pp}H_{qq} - (H_{pq})^2 > 0$ implies

$$-c_{qq}(2 - ff_{pp}/f_p^2) < 2f_{qq}/f_p + [2f_{pp}f_q^2 - 4f_pf_qf_{pq} + ff_{pq}^2 - ff_{pp}f_{qq}]/f_p^3. \tag{10}$$

Now we shall state some immediate consequences of the optimality conditions (4)–(10). From (6) and (7), we easily obtain

$$c_q = -f_q/f_p = (\mu/\tau)(p/q), \tag{11}$$

where $\mu = f_q q/f$ is the quality elasticity of demand, and $\tau = -f_p p/f$ is the price elasticity of demand. A simple economic interpretation of (11) is that the firm produces at the quality level where the cost of quality increase equals the price increase that consumers are willing to pay. Note that the above economic interpretation of (11) can also be seen in Dorfman and Steiner (1954), as well as in Tellis and Wernerfelt (1987).

Integrating (4) with condition (5), we have the future benefit of having one more unit produced

$$\alpha(t) = \int_t^T [(p - c + \alpha) f_x - c_x f] e^{-rs} \, ds. \tag{12}$$

3. Subclasses of the general formulation

While the general formulation discussed in the above section is useful to gain insight into the factors affecting the optimal price and quality, we can obtain stronger results for specific cases. First, we consider the case where sales depend on price and quality, but not word of mouth. Next, we discuss the case where demand functions are separable.

3.1. Static demand functions

In this subsection we will study the case where sales are a function of price and quality. This situation characterizes the case where word-of-mouth effect is not important, e.g. inexpensive or established products.

In this special case, (12) can be reduced to the following:

$$\alpha(t) = -\int_t^T e^{-rs} c_x f \, ds > 0 \quad \text{for all } t < T, \tag{13}$$

which represents that the future benefit of having one more unit sold is always positive (due to the learning curve effect), if there is no saturation effect on demand. By taking the time derivatives of the optimal price and the optimal quality from (6) and (11), we obtain (see (A1)–(A10) in Appendix A for details):

$$\dot{p} = \frac{d_1(H_{qq} + c_q H_{pq})}{D}, \qquad \dot{q} = -\frac{d_1(H_{qp} + c_q H_{qp})}{D}, \tag{14}$$

where $d_1 = -r\alpha f_p > 0$, and $D = H_{pp}H_{qq} - (H_{pq})^2 > 0$.

It is clear from (14) that if the discount rate r is zero (thus, $d_1 = 0$) the problem reduces to a static problem, and both price and quality stay the same over the entire planning horizon. On the other hand, price and quality cannot be constant for any period of time if the discount rate is non-zero. For a positive discount rate, the results are characterized as follows.

Theorem 1. *If $r > 0$ and $f = f(p,q)$, then:*

	Conditions	*Results*
Case 1.	$H_{qq} + c_q H_{pq} > 0$	$\dot{p} > 0$ *and* $\dot{q} > 0$
Case 2.	$H_{qp} + c_q H_{pp} > 0$	$\dot{p} < 0$ *and* $\dot{q} < 0$
Case 3.	$H_{pq} < 0$	$\dot{p} < 0$ *and* $\dot{q} > 0$

Proof. See Appendix A.

In Case 1, $H_{qq} + c_q H_{pq} > 0$ implies $H_{qp} > 0$. Similarly, $H_{qp} + c_q H_{pp} > 0$ in Case 2 also indicates $H_{qp} > 0$. In fact, $H_{pq} > 0$ means the total profit H (at time t) will be increased by increasing (or decreasing) both price and quality simultaneously. In this case, $r > 0$ and $f = f(p,q)$, the future benefit of having one more unit sold α is always positive. Thus, the firm wants to subsidize early units in order to increase production (or sales) now and reduce costs (by learning effect) later on. The firm's subsidization of early units is greatest in the zero discount case. As the discount rate increases, the firm behaves increasingly myopically, and reduces the degree of early subsidization. The subsidization can be either in price, or in quality, depending on which one is more efficient in increasing sales and reducing costs, which in turn depends on $H_{qq} + c_q H_{pq} > 0$ or $H_{qp} + c_q H_{pp} > 0$.

The results in Theorem 1 can be interpreted to mean that price is more efficient than quality in increasing sales and reducing costs, provided $H_{qq} + c_q H_{pq} > 0$. Thus, the optimal price is initially low to stimulate early buyers and then goes higher, and the optimal quality follows the same pattern because $H_{pq} > 0$. On the other hand, if $H_{qp} + c_q H_{pp} > 0$, then increasing quality is more efficient in stimulating early buyers than reducing price, which implies that the optimal quality is initially high and then goes down, and the optimal price follows the same pattern. Finally, in Case 3, $H_{pq} < 0$ indicates the firm should adopt a policy that price and quality go up or down in opposite directions. In this case the optimal strategy is to lower price and increase quality over time. Otherwise, the firm may go out of business soon by charging high price for a low quality product.

Note that this theorem is a generalization of Theorem 1 of Kalish (1983), in which quality is not a decision variable, and hence optimal price decreases monotonically over time. The two equations in (14) replace the single equation (16) on page 140 of Kalish (1983), which accounts for the more complicated statement and the above interpretations of our Theorem 1 as compared to Theorem 1 on page 141 of Kalish.

Theorem 1 also implies that if $r > 0$ and $f = f(p,q)$, then an increase in optimal price must be accompanied by an increase in product quality. However, the reverse does not hold. In other words, an increase in quality does not imply a price increase. For a counterexample, see Case 3 of Theorem 1.

To illustrate the above mathematical results, we now apply Theorem 1 to solve the following two simple examples.

Example 1. Linear price–quality demand function.

Assume that the sales rate f is as follows:

$$f(p,q) = a_0 - a_1 p + a_2 q + a_3(q/p), \tag{15}$$

where a_0, a_1, a_2, and $a_3 \geq 0$. It is easy to see that $f_p = -a_0 - a_3 q/p^2 < 0$, $f_q = a_2 + a_3/p > 0$, $f_{pq} = -a_3/p^2 < 0$, $f_{pp} = 2a_3 q/p^3 > 0$, and $f_{qq} = 0$. From (A6) in Appendix A, we have:

$$H_{qq} + c_q H_{pq} = f\left[-c_{qq} + (f_{pq} f_q - f_{qq} f_p)/f_p^2\right] = -f a_3(a_2 p + a_3)/p^3 < 0. \tag{16}$$

Consequently, only Cases 2 and 3 of Theorem 1 can occur in this example.

If $a_0 = a_1 = 0$, then from (A5) we obtain $H_{qp} + c_q H_{pp} > 0$. This implies that increasing quality is more efficient in stimulating early buyers than reducing price. Therefore, the optimal quality is initially high and then goes down, and the optimal price follows the same pattern. On the other hand, if $a_0 > 4a_1 p$, $a_1 p > a_3(q/p)$, and $a_2 = 0$, then from (A4) we know $H_{pq} < 0$. Hence, the optimal quality is increasing while the optimal price is declining.

Example 2. Simple price–quality demand function.

Assume consumer types are distributed uniformly on the interval $[0, M]$, and a type s consumer is willing to pay up to sq for a unit of product with quality q. The market for the product of quality q at price p is thus defined as the set of consumer types who are willing to pay the amount which is greater than or equal to the price. That is,

$$S = \{s \in [0, M] \mid sq - p \geq 0\} = [p/q, M]. \tag{17}$$

Therefore, the demand function becomes

$$f(p,q) = M - p/q, \tag{18}$$

which has been used in many of the previous studies, such as Mussa and Rosen (1978) and Moorthy (1988). Differentiating f, we have $f_p = -1/q$, $f_q = p/q^2$, $f_{pq} = 1/q^2$, $f_{pp} = 0$, and $f_{qq} = -2p/q^3$. From (A5), we obtain

$$H_{qp} + c_q H_{pp} = f(f_{pp} f_q - f_{qp} f_p)/f_p^2 = f/q > 0. \tag{19}$$

By applying Theorem 1, we know that both optimal price and quality are decreasing simultaneously.

3.2. Separable demand functions

We shall now investigate the case where the interaction among price, quality and cumulative sales takes one of the following separable forms: (1) sales function $f(p,q,x) = h(q,x)P(p)$, (2) $f = k(p,q)X(x)$, or (3)

$f = w(p,x)Q(q)$. However, we will examine only the first two cases because the third one does not provide any results of interest. These separable sales functions are interesting for two reasons: (1) they are naturally simple ways to model, and they are also mathematically tractable, and (2) most of the previous studies on pricing for new product diffusion models have used such separable demand functions, in which product quality is not a factor.

3.2.1. Separable demand functions in price

Several articles that examined pricing over diffusion models have adopted the function $f(p,x) = X(x)\exp(-gp)$, where $\exp(y) = e^y$ and g is a positive number. For example, see Kalish (1983) and Thompson and Teng (1984). Hence, we assume here that $f(p,q,x) = h(q,x)\exp(-gp)$. In this case, we can simplify the system of equations (A1) and (A2) in Appendix A that characterizes the optimal price and quality to the following:

$$\dot{p} - \left(\frac{h_q}{gh}\right)\dot{q} = -r\alpha + \exp(-gp)h_x/g, \tag{20}$$

$$\left(\frac{h_{qq}h - h_q^2}{gh^2} - c_{qq}\right)\dot{q} = \exp(-gp)\frac{h_x h_q - h_{qx}h}{gh}. \tag{21}$$

From (20), we obtain $H_{pq}/f_p = -h_q/(gh)$. Therefore, $H_{pq} > 0$ and the optimal price and quality must go up or down at the same time.

Theorem 2. *If $f = h(q,x)\exp(-gp)$, then an optimal price–quality policy is characterized by the following:*

(1) *Optimal quality is increasing at time t if $h_x h_q < h_{qx}h$ at time t, and decreasing if $h_x h_q > h_{qx}h$ at time t.*

(2) *For a zero discount rate, optimal price is decreasing whenever $h_x < 0$ and $h_x h_q \geq h_{qx}h$, and increasing whenever $h_x > 0$ and $h_x h_q \leq h_{qx}h$.*

Proof. See Appendix A.

The results of Theorem 2 can be given the following useful interpretations:

1. The condition $h_x h_q < h_{qx}h$ is equivalent to $\mu_x > 0$ where μ is the quality elasticity of demand ($\mu = f_q q/f$, i.e., how much demand changes as the quality of the product is changed by one percent). With this observation at hand the first part in Theorem 2 states that if demand becomes more quality elastic as we move through the stages of the product life cycle, then quality should be increased, and vice versa.

2. The condition $h_x > 0$ and $h_x h_q \leq h_{qx}h$ means that demand will increase with market penetration (i.e., $h_x > 0$), and the optimal policy for the firm is to increase its product quality (because of $h_x h_q \leq h_{qx}h$). Since $H_{pq} > 0$, the firm should also increase its price to compensate for a heavy increase in spending on quality. On the other hand, if the market is saturated, then future demand decreases with additional sales now (i.e., $h_x < 0$), and is negative quality elastic (i.e., $h_x h_q \geq h_{qx}h$). Hence, the policy implication is that one should lower product quality and hence lower the selling price in order to counterbalance the loss of sales resulting from a decrease in product quality and saturation effects on demand.

Note that for a positive discount rate the problem seems too complicated to yield a clear and decisive result on pricing. For illustration, we shall apply the theoretical results to a straightforward diffusion model.

Example 3. Assume that the marginal sales volume f is proportional to the quality level q as follows:

$$\begin{aligned} h(q,x) &= (e_1 + e_2 q)(M - x) + (e_3 + e_4 q)(M - x)x \\ &= (\text{Quality effects on innovators}) + (\text{Quality effects on imitators}). \end{aligned} \tag{22}$$

Rewriting, we can give another interpretation:

$$h(q,x) = (M-x)(e_1 + e_2 q + e_3 x + e_4 qx)$$
$$= (\text{Saturation effect}) \times (\text{Main effects} + \text{Interaction effect}), \qquad (22')$$

where e_1, e_2, e_3, and e_4 are nonnegative constants. Differentiating $h(q,x)$, and after some algebraic manipulations, we can get

$$h_x h_q - h_{qx} h = (M-x)^2 (e_2 e_3 - e_1 e_4). \qquad (23)$$

It is easy to see from Theorem 2 that the optimal quality is increasing if $e_1 e_4 > e_2 e_3$, and decreasing if $e_1 e_4 < e_2 e_3$.

A simple managerial interpretation of the result is as follows: If the relative effect of quality on the innovators is higher than that on the imitators (i.e., $e_2/e_1 < e_4/e_3$, or $e_1 e_4 > e_2 e_3$), then the optimal quality is initially high to stimulate early buyers (namely, the innovators) and then goes lower. Otherwise (i.e., $e_2/e_1 > e_4/e_3$) the optimal quality increases as the number of adopters increases.

Note that a similar result on optimal advertising expenditure, but not on quality, can be seen in Corollary 1 of Dockner and Jorgensen (1988a) and in Theorem 2 of Teng and Thompson (1983). Similarly, applying Theorem 2, we know that for a zero discount rate and $e_1 e_4 = e_2 e_3$ the optimal price is increasing during an initial time, then is decreasing when the market becomes half saturated, i.e. $x \geq M/2$.

3.2.2. Separable demand functions in cumulative sales

When $f = k(p,q)X(x)$, the system of equations that characterizes the optimal solution reduces as follows:

$$H_{pp}\dot{p} + H_{pq}\dot{q} = -r\alpha k_p X - k^2 X' X, \qquad (24)$$

$$(H_{qp} + c_q H_{pp})\dot{p} + (H_{qq} + c_q H_{pq})\dot{q} = 0. \qquad (25)$$

It is clear from (25) that optimal price is constant on any interval if, and only if, optimal quality remains the same on the same interval.

Firstly, we shall consider the case of a zero discount rate. A zero discount rate ($r = 0$) means that the firm is indifferent between having one dollar today or one dollar in the near future. Although a zero discount rate may be somewhat unrealistic, the case can serve as an approximation in situations where the discount rate is relatively low. The following result holds for this case:

Theorem 3. *For a zero discount rate and $f = k(p,q)X(x)$, we have:*

Conditions	*Diffusion effect* $X' > 0$	*Saturation effect* $X' < 0$
$H_{qq} + c_q H_{pq} > 0$	$\dot{p} < 0$ *and* $\dot{q} < 0$	$\dot{p} > 0$ *and* $\dot{q} > 0$
$H_{qp} + c_q H_{pp} > 0$	$\dot{p} > 0$ *and* $\dot{q} > 0$	$\dot{p} < 0$ *and* $\dot{q} < 0$
$H_{pq} < 0$	$\dot{p} > 0$ *and* $\dot{q} < 0$	$\dot{p} < 0$ *and* $\dot{q} > 0$

Proof. It is obvious from (24), (25), and Lemmas 1–3 of Appendix A.

If there is no diffusion effect ($f_x = 0$) and the discount rate is positive (i.e., having one dollar today is more valuable than having one dollar in the near future), then the optimal policy for the firm is to trade more profits currently for less profits in the future. On the other hand, if there is a diffusion effect and a zero discount rate, then the firm should accept less profits now in order to stimulate early adopters, which in turn will stimulate more demand and profits in the future due to the diffusion effect (or positive word-of-mouth effect). Therefore, comparing Theorem 1 with Theorem 3, we can see that the optimal policies for those two cases are completely opposite. Using a similar argument, if there is a saturation effect ($f_x < 0$), then the optimal policy for the firm is to generate more profits currently, before the market becomes oversaturated. Hence, the optimal policy in this case is the same as that described in Theorem 1.

Secondly, for a positive discount rate the case with a learning curve effect is apparently intractable, or has no clear results. We hence assume there is no learning effect on production cost, $c = c(q)$. Now, we are ready to state the results for the case where the demand function is concave in x ($X'' < 0$), e.g. the standard Bass model, and there is no learning effect ($c = c(q)$). For convenience, we from now on use Cases 1, 2, and 3 to represent the corresponding conditions of $H_{qq} + c_q H_{pq} > 0$, $H_{qp} + c_q H_{pp} > 0$, and $H_{pq} < 0$, respectively.

Theorem 4. *If $f = k(p,q)X(x)$ with $X'' < 0$ and $c = c(q)$, then optimal price and optimal quality are characterized as follows:*

Case	*Diffusion effect* $X'(x(T)) > 0$	*Diffusion then Saturation* $X'(x(0)) > 0$ *and* $X'(x(T)) < 0$	*Saturation effect* $X'(x(0)) < 0$
1	$\dot p$ *and* $\dot q < 0$ *for any t*	*first* $\dot p$ *and* $\dot q < 0$ *then* $\dot p$ *and* $\dot q > 0$	$\dot p, \dot q < 0$ *then* $\dot p, \dot q > 0$ *or* $\dot p, \dot q > 0$ *for any t*
2	$\dot p$ *and* $\dot q > 0$ *for all t*	*first* $\dot p$ *and* $\dot q > 0$ *then* $\dot p$ *and* $\dot q < 0$	$\dot p, \dot q > 0$ *then* $\dot p, \dot q < 0$ *or* $\dot p, \dot q < 0$ *for all t*
3	$\dot p > 0$ *and* $\dot q < 0$ *for any t*	*first* $\dot p > 0$ *and* $\dot q < 0$ *then* $\dot p < 0$ *and* $\dot q > 0$	$\dot p > 0$, $\dot q < 0$ *then* $\dot p < 0$, $\dot q > 0$ *or* $\dot p < 0$, $\dot q > 0$ *for any t*

Proof. See Appendix A.

Note that $X'(x(T)) > 0$ with $X'' < 0$ implies $X' > 0$ over the entire planning horizon. Thus, the optimal strategies in this case are the same as those described in Theorem 3 with $X' > 0$. Similarly, $X'(x(0)) < 0$ with $X'' < 0$ means that $X' < 0$ over the entire planning time horizon, and hence optimal policies are similar to those of Theorem 3 with $X' < 0$. Finally, $X'(x(0)) > 0$ and $X'(x(T)) < 0$ implies $X' > 0$ first then $X' < 0$. Therefore, the optimal solution to this case can also be obtained from Theorem 3 with $X' > 0$ first, then $X' < 0$.

Finally, for the case where there is a learning effect on production cost, and the demand function is convex in x (i.e., $X'' > 0$), we can characterize the optimal solution by:

Theorem 5. *If $f = k(p,q)X(x)$ with $X'' > 0$, then the optimal price and optimal quality are characterized as follows:*

Case	*Saturation effect* $X'(x(T)) < 0$	*Saturation then Diffusion* $X'(x(0)) < 0$ *and* $X'(x(T)) > 0$	*Diffusion effect* $X'(x(0)) > 0$
1	$\dot p$ *and* $\dot q > 0$ *for any t*	*first* $\dot p$ *and* $\dot q > 0$ *then* $\dot p$ *and* $\dot q < 0$	$\dot p, \dot q > 0$ *then* $\dot p, \dot q < 0$ *or* $\dot p, \dot q < 0$ *for any t*
2	$\dot p$ *and* $\dot q < 0$ *for all t*	*first* $\dot p$ *and* $\dot q < 0$ *then* $\dot p$ *and* $\dot q > 0$	$\dot p, \dot q < 0$ *then* $\dot p, \dot q > 0$ *or* $\dot p, \dot q > 0$ *for all t*
3	$\dot p < 0$ *and* $\dot q > 0$ *for any t*	*first* $\dot p < 0$ *and* $\dot q > 0$ *then* $\dot p > 0$ *and* $\dot q < 0$	$\dot p < 0$, $\dot q > 0$ *then* $\dot p > 0$, $\dot q < 0$ *or* $\dot p > 0$, $\dot q < 0$ *for any t*

Proof. See Appendix A.

The results given in Theorem 5 with $X'(x(0)) < 0$ and $X'(x(T)) > 0$ are the less interesting cases, since in many situations there will not be a positive diffusion effect on demand after the market is saturated (e.g., the Bass model). Note that our Theorems 3, 4, and 5 are general forms of the corresponding Kalish's Theorems 2, 3, and 4, respectively.

4. Conclusions

In this paper, we have extended Kalish's monopolistic pricing results (Kalish, 1983) to models in which quality as well as price is a decision variable. We then have established a complete analysis of optimal inter-relationships among price, quality, learning effect, and diffusion demand. We assume here that consumers can easily identify product quality. From (6), we learn that the optimal quality level q^* (i.e., $c'(q^*) = -f_q/f_p > 0$) is always higher than the minimum cost quality level q_0 (i.e., $c'(q_0) = 0$). In contrast, Shapiro (1982) assumes that consumers can not observe product quality before purchasing. Consequently, he concludes that the optimal quality is below the minimum cost quality level q_0.

The theoretical results obtained here confirm and extend prior results in the literature. The above Theorems tell us that there are only three quantitatively different kinds of optimal price–quality policies as follows: (A) Both optimal price and quality are declining simultaneously; (B) Optimal price is declining while optimal quality is going higher; (C) Both optimal price and quality are going higher at the same time. The first case of this result is demonstrated in the IBM personal computers product line (Moorthy and Png, 1992). ''On 26 June 1990, IBM released its PS/1 home computer, several years after its PS/2 business-oriented microcomputer.'' The second case is demonstrated in the Hewlett-Packard Laser Printers product line (Ghosh and Narasimhan, 1993). ''HP has continued to hold a large share of the laser printer market since its introduction of the LaserJet printer. It has followed that with several more technologically advanced versions of the LaserJet printer. With each model, it has improved overall quality, and simultaneously reduced list price.'' In contrast to HP, the microprocessor-industry dominating firm, Intel Corporation, sequentially updates its microprocessor families (from 286 chips, then 386 and 486 chips, and now Pentium chips), and improves prices for its new generation products. This may serve as an example for the third case. The other theoretical results also show the following managerial phenomena: (1) In a market where word of mouth is unimportant, higher prices must be preceded by higher quality. However, lower prices in this market do not have to be preceded by lower quality. (2) If there is diffusion effect on demand, then it is possible that higher prices may go along with either lower or higher quality. Many other intuitively reasonable managerial results can be obtained from the above theorems.

The model can be extended in several ways. For instance, we can extend the monopolistic model to a competitive duopolistic or oligopolistic market. Also, we can consider a more general model in which advertising is also a control variable. Finally, we can study product warranty as to quality in a market where consumers can not observe product quality before purchasing.

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Appendix A. Optimal solution characterization and proofs of lemmas and theorems

In order to analyze the relationship between price and quality, we take the time derivative of the optimal price given by (6), substitute (4) for $\dot{\alpha}$, and (11) for c_q, and rearrange terms. The result is:

$$\left(2 - ff_{pp}/f_p^2\right)\dot{p} + \left[\left(2f_pf_q - f_{pq}f\right)/f_p^2\right]\dot{q} = -r\alpha - 2ff_x/f_p + f_{px}f^2/f_p^2. \tag{A1}$$

Similarly, taking the derivative of the optimal quality given by (11) with respect to time t, and rearranging terms, we derive

$$\left[\left(f_{pp}f_q - f_{qp}f_p\right)/f_p^2\right]\dot{p} + \left[\left(f_{pq}f_q - f_{qq}f_p\right)/f_p^2 - c_{qq}\right]\dot{q} = f\left(f_{qx}f_p - f_{px}f_q\right)/f_p^2. \tag{A2}$$

For simplicity, we let

$$d_1 = -r\alpha f_p - 2ff_x + f_{px}f^2/f_p, \quad d_2 = f^2\left(f_{qx}f_p - f_{px}f_q\right)/f_p^2, \quad d_3 = d_2 - c_qd_1. \tag{A3}$$

Since

$$H_{pp}/f_p = 2 - \left(ff_{pp}\right)/f_p^2, \qquad H_{pq}/f_p = \left(2f_pf_q - f_{pq}f\right)/f_p^2, \tag{A4}$$

$$\left[H_{qp} + c_qH_{pp}\right]/f = \left(f_{pp}f_q - f_{qp}f_p\right)/f_p^2, \tag{A5}$$

$$\left[H_{qq} + c_qH_{pq}\right]/f = -c_{qq} + \left(f_{pq}f_q - f_{qq}f_p\right)/f_p^2, \tag{A6}$$

We can simplify (A1) and (A2) as the following system of equations:

$$H_{pp}\dot{p} + H_{pq}\dot{q} = d_1, \qquad \left[H_{qp} + c_qH_{pp}\right]\dot{p} + \left[H_{qq} + c_qH_{pq}\right]\dot{q} = d_2. \tag{A7}$$

or

$$H_{pp}\dot{p} + H_{pq}\dot{q} = d_1, \qquad H_{qp}\dot{p} + H_{qq}\dot{q} = d_2 - c_qd_1 = d_3. \tag{A7$'$}$$

Solving equations in (A7) or (A7′), we obtain

$$\dot{p} = \left(d_1H_{qq} - d_3H_{pq}\right)/D, \qquad \dot{q} = \left(d_3H_{pp} - d_1H_{qp}\right)/D, \tag{A8}$$

where $D = H_{pp}H_{qq} - (H_{pq})^2 > 0$.

Lemma 1. *If $H_{qq} + c_qH_{pq} > 0$, then $H_{pq} > 0$ and $H_{qp} + c_qH_{pp} < 0$.*

Proof. From $H_{qq} < 0$ and $c_q > 0$, we know that $H_{qq} + c_qH_{pq} > 0$ implies $H_{pq} > 0$. Since the determinant

$$\begin{vmatrix} H_{pp} & H_{pq} \\ H_{qp} + c_qH_{pp} & H_{qq} + c_qH_{pq} \end{vmatrix} = \begin{vmatrix} H_{pp} & H_{pq} \\ H_{qp} & H_{qq} \end{vmatrix} = D > 0, \tag{A9}$$

and $H_{pp} < 0$, we obtain $H_{qp} + c_qH_{pp} < 0$ if $H_{qq} + c_qH_{pq} > 0$. □

Lemma 2. *If $H_{qp} + c_qH_{pp} > 0$, then $H_{pq} > 0$ and $H_{qq} + c_qH_{pq} < 0$.*

Proof. It is similar to Lemma 1.

Lemma 3. *If $H_{pq} = H_{qp} < 0$, then $H_{qp} + c_qH_{pp} < 0$ and $H_{qq} + c_qH_{pq} < 0$.*

Proof. It immediately follows from Lemmas 1 and 2.

Proof of Theorem 1. In this case, $d_1 = -r\alpha f_p > 0$, $d_2 = 0$ and $d_3 = -c_q d_1$. (A8) reduces to:

$$\dot{p} = d_1(H_{qq} + c_q H_{pq})/D, \qquad \dot{q} = -d_1(H_{qp} + c_q H_{qp})/D. \tag{A10}$$

From Lemmas 1 through 3, and (A10), we can easily obtain this theorem. □

Proof of Theorem 2. From (A1), (A2) and (A9), we obtain $(h_{qq}h - h_q^2)/(g_h^2) - c_{qq} = D/(ff_p) < 0$. Consequently, (21) implies that $\dot{q}$ and $h_x h_q - h_{qx}h$ have the opposite signs. This proves Part 1 of Theorem 2.

For a zero discount rate ($r = 0$), by (20) we know that

$$\dot{p} = \frac{h_x \exp(-gp)}{g} + \frac{h_q \exp(-gp)(h_x h_q - h_{qx} h)}{(gh)^2 D}. \tag{A11}$$

Hence, $\dot{p} < 0$ when $h_x < 0$ and $h_x h_q - h_{qx} h > 0$, and $\dot{p} > 0$ when $h_x > 0$ and $h_x h_q - h_{qx} h < 0$. □

Proof of Theorem 4 (The proof is similar to that of Kalish (1983)). Taking the derivatives of (24) and (25), respectively, and substituting $\dot{p} = 0$ and $\dot{q} = 0$ (by (25), $\dot{p} = 0$ if, and only if $\dot{q} = 0$), we get the second derivatives (denoted by ″) of the optimal path at such point are:

$$\left(2 - \frac{kk_{pp}}{k_p^2}\right)p'' + \left(\frac{2k_p k_q - k_{pq} k}{k_p^2}\right)q'' = -\frac{k^2 X'' f}{k_p}, \tag{A12}$$

and

$$\left(\frac{k_{pp} k_q - k_{qp} k_p}{k_p^2}\right)p'' + \left(\frac{k_{pq} k_q - k_{qq} k_p}{k_p^2} - c_{qq}\right)q'' = 0. \tag{A13}$$

Thus,

$$p'' = -\left(\frac{k^2 X'' f}{k_p}\right)\frac{H_{qq} + c_q H_{pq}}{D k_p X}, \tag{A14}$$

$$q'' = \left(\frac{k^2 X'' f}{k_p}\right)\frac{H_{qp} + c_q H_{pp}}{D k_p X}, \tag{A15}$$

when $\dot{p} = \dot{q} = 0$. Similarly, from (24), (25), and $\alpha(t) = 0$, we have

$$\dot{p}(T) = -\left(\frac{k^2 X'}{k_p}\right)\frac{H_{qq} + c_q H_{pq}}{D k_p X}, \tag{A16}$$

and

$$\dot{q}(T) = \left(\frac{k^2 X'}{k_p}\right)\frac{H_{qp} + c_q H_{pp}}{D k_p X}. \tag{A17}$$

For $X'(x(T) > 0$, if $H_{qq} + c_q H_{pq} > 0$ then $\dot{p}(T) < 0$ by (A16). Assuming twice differentiability of $p(t)$, if $p(t)$ changes signs at $t = s$ ($\dot{p}(s) > 0$), then from (A14) we have $p''(s) > 0$. It leads to $\dot{p}(T) > 0$, which contradicts to $\dot{p}(T) < 0$. Therefore, $\dot{p}(t) < 0$ for any t if $H_{qq} + c_q H_{pq} > 0$, and vice versa. By a similar argument, we can easily prove Cases 2 and 3.

$X'(x(0)) > 0$ and $X'(x(T)) < 0$ imply that there exists an s such that $X'(x(s)) = 0$, and $X'(x(t)) < 0$ for any $t > s$. Consequently,

$$\alpha(s) = \int_s^T e^{-rt}\left(-k^2X'/k_p\right)dt < 0. \tag{A18}$$

By using (24) and (25), we have

$$\dot{p}(s) = -r\alpha\left(H_{qq} + c_q H_{pq}\right)/\left(Dk_p X\right), \tag{A19}$$

and

$$\dot{q}(s) = r\alpha\left(H_{qp} + c_q H_{pp}\right)/\left(Dk_p X\right). \tag{A20}$$

$H_{qq} + c_q H_{pq} > 0$ immediately follows $\dot{p}(s) < 0$. Using an analogous argument of the proof of the case in which $X'(x(T)) > 0$, we get $\dot{p}(t) < 0$ for all $t < s$. From (A16) and $X'(x(T)) < 0$, we know $\dot{p}(T) > 0$. Since $\dot{p}(s) < 0$, $\dot{p}(T) > 0$, and $p''(t) > 0$ at the point that $\dot{p}(t) = 0$, we know that $\dot{p}(t)$ cannot change signs more than once between s and T. This completes the proof that optimal price is decreasing and then increasing when $H_{qq} + c_q H_{pq} > 0$. Again, by an analogous argument, we can easily prove Cases 2 and 3 when $X'(x(0)) > 0$ and $X'(x(T)) < 0$.

Finally, $X'(x(0)) < 0$ and $X''(x(t)) < 0$ for any t imply $X'(x(t)) < 0$ for any t. If $H_{qq} + c_q H_{pq} > 0$, $\dot{p}(T) < 0$ follows quickly from (A16). However, one change of the sign of $\dot{p}$ cannot be ruled out. Using a similar argument, we can show the other cases in which $H_{qp} + c_q H_{pp} > 0$ or $H_{qp} < 0$. □

Proof of Theorem 5. In this case we get that the second derivative of price at $\dot{p} = 0$ is:

$$\left(2 - \frac{kk_{pp}}{k_p^2}\right)\dot{p} + \left(\frac{2k_p k_q - k_{pq}k}{k_p^2}\right)\dot{q} = -\frac{k^2X''f}{k_p} - rc_x f.$$

By using a similar argument as in the proof of Theorem 4 we can easily prove this theorem.

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